SERIES REPRESENTATIONS FOR MULTIVARIATE GENERALIZED GAMMA PROCESSES VIA A SCALE INVARIANCE PRINCIPLE

Hemant Ishwaran and Mahmoud Zarepour

Cleveland Clinic and University of Ottawa

Supplementary Material

This note contains proofs for Theorems 1, 3, and 4.

Proof of Theorem 1. It is clear that \( \sum_{i=1}^{\infty} \varepsilon_{\Gamma_i}() \) is a Poisson random measure with mean measure \( \lambda \), where \( \lambda \) is Lebesgue measure. Use \( \text{PRM}(\lambda) \) to denote this. From Proposition 3.8 of Resnick (1987),

\[
\sum_{i=1}^{\infty} \varepsilon_{(\Gamma_i, U_i, V_i)}(\cdot)
\]

is a \( \text{PRM}(d\nu) \) where \( d\nu = d\lambda \times dF \) and \( F \) is the joint distribution for \((U_1, V_1)\). Therefore, from Proposition 3.7 of Resnick (1987), the point process

\[
\xi(\cdot) = \sum_{i=1}^{\infty} \varepsilon_{(N^{-1}(\Gamma_i), N^{-1}(\Gamma_i V_i))}(\cdot)
\]

is a \( \text{PRM}(\Pi) \) for \( \Pi = \nu \circ T^{-1} \), where

\[
T(x, y, z) = \left( N^{-1}(xy), N^{-1}(xz) \right).
\]

We have

\[
\nu \circ T^{-1}((a, \infty) \times (b, \infty)) = \nu \left\{ (x, y, z) : N^{-1}(xy) > a \text{ and } N^{-1}(xz) > b \right\}
\]

\[
= \nu \left\{ (x, y, z) : xy < N(a) \text{ and } xz < N(b) \right\}
\]

\[
= \nu \left\{ (x, y, z) : x < (N(a)/y) \wedge (N(b)/z) \right\}
\]

\[
= \int_0^{\infty} \int_0^{\infty} \int_0^{N(a)/y \wedge N(b)/z} dt \, F(dy, dz)
\]

\[
= \mathbb{E} \left( \frac{N(a)}{U_1} \wedge \frac{N(b)}{V_1} \right).
\]

Part (ii) can be verified as in part (i). For part (iii), use part (ii), and observe that

\[
\int x \sum_{i=1}^{\infty} \varepsilon_{(N^{-1}(\Gamma_i V_i), X_i)}(dx \times \cdot) \quad \overset{\not\sim}{=} \quad \int x \sum_{i=1}^{\infty} \varepsilon_{(N^{-1}(\Gamma_i h), X_i)}(dx \times \cdot)
\]

\[
= \sum_{i=1}^{\infty} N^{-1}(\Gamma_i h) \varepsilon_{X_i}(\cdot).
\]
Proof of Theorem 3. The first limit in part (i) follows using Proposition 3.21 of Resnick (1987) and (7). For the second part of (i) we mimic the proof of Theorem 4 of Resnick and Greenwood (1979). Observe that the map

\[ T_h \left( \sum_k \varepsilon_{(t_k, y_k)}(\cdot) \right) = \sum_{t_k \leq t} y_k I\{y_k > h\} \]

defined on the set of point processes on \([0, 1] \times \mathbb{R}^+\) to \(D[0, 1]\) is continuous (there are a finite number of terms in the summation). Therefore, for \(h > 0\),

\[ \sum_{i=1}^{nt} Z_{i,n} I\{Z_{i,n} > h\} \xrightarrow{d} \sum_{i=1}^{\infty} M_{\alpha,\delta,\theta}^{-1}(\Gamma_i) I\{U_i \leq t\} I\{M_{\alpha,\delta,\theta}^{-1}(\Gamma_i) > h\} \]

in \(D[0, 1]\). Let \(d(\cdot, \cdot)\) be the Skorohod metric on \(D[0, 1]\). Then,

\[
\mathbb{P}\left\{ d\left( \sum_{i=1}^{\lfloor nt \rfloor} Z_{i,n}; \sum_{i=1}^{\lfloor nt \rfloor} Z_{i,n} I\{Z_{i,n} > h\} \right) > \epsilon \right\} \\
\leq \mathbb{P}\left\{ \sup_{k \leq n} \sum_{i=1}^{k} Z_{i,n} I\{Z_{i,n} \leq h\} > \epsilon \right\} \\
\leq \mathbb{P}\left\{ \sum_{i=1}^{n} Z_{i,n} I\{Z_{i,n} \leq h\} > \epsilon \right\} \\
\leq \mathbb{E}\left( Z_{1,n} I\{Z_{1,n} \leq h\} \right)^{-1} n \mathbb{P}\{Z_{1,n} \in dx\} \\
= \mathbb{E}^{-1} \int_{0}^{h} x \mathbb{P}\{Z_{1,n} \in dx\} \\
\to \mathbb{E}^{-1} \int_{0}^{h} \frac{\delta}{\Gamma(1 - \alpha)} x^{-\alpha+1} \exp(-\theta x) dx,
\]

as \(n \to \infty\). Observe that the right-hand side goes to zero as \(h \downarrow 0\).

Part (ii) follows from part (i). \(\square\)

Proof of Theorem 4. By Bayes Theorem,

\[
\int \int g(v, \mu) Q_n^* (dv, d\mu) = \frac{\int \int g(v, \mu) L(v) Q_n (dv, d\mu)}{\int \int L(v) Q_n (dv, d\mu)}. \quad \text{(S1)}
\]

Consider the numerator on the right-hand side. By definition, this equals

\[
\int \int g(v, \mu) \left( \prod_s \prod_{j=1}^{d} \psi_{s,j}(v_{s,j}) \mu_j(dv_{s,j}) \right) G_n(d\mu) \\
= \int \int g(v, \mu) \left( \prod_s \prod_{j=1}^{d} \psi_{s,j}(v_{s,j}) \left\{ \sum_{i=1}^{n} Z_i \varepsilon X_i (dv_{s,j}) \right\} \right) F_n(dZ) P_0^n(dX),
\]

where \(F_n(dZ)\) is the joint distribution for \(Z = (Z_1, \ldots, Z_n)\). Let \(Z_0 = \sum_{i=1}^{n} Z_i\). Then \(Z_0\) has a gamma distribution with shape parameter \(\alpha\) and scale parameter \(\beta = 1\). Furthermore,
$Z_0$ is independent of $p = (p_1, \ldots, p_n)$, where $p_i = Z_i / Z_0$. Rewriting $Z_i$ as $Z_0 \times (Z_i / Z_0)$, deduce that the right-hand side of the previous expression can be rewritten as

$$\alpha \iiint g(v, \mu) \left( \prod_s \prod_{j=1}^d \psi_{s,j}(v_{s,j}) \left\{ \sum_{i=1}^n p_i \epsilon_i X_i (dv_{s,j}) \right\} \right) \pi_n (dp) P_0^n (dX). \quad \text{(S2)}$$

Define conditionally independent variables $K_{s,j}$ such that

$$\mathbb{P}\{K_{s,j} \in \cdot | p\} = \sum_{i=1}^n p_i \epsilon_i (\cdot).$$

Because $P_0$ is non-atomic, it follows that $v_{s,j} = X_i$ in (S2) if and only if $K_{s,j} = i$. Consequently (S2) becomes

$$\alpha \iiint g^*(v, \mu) \left( \prod_s \prod_{j=1}^d \psi_{s,j}(v^*_{s,j}) \left\{ \sum_{i=1}^n p_i \epsilon_i (dK_{s,j}) \right\} \right) \pi_n (dp) P_0^n (dX).$$

Apply the same argument to the denominator of (S1). Note the cancellation of $\alpha$. \qed